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STRATEGIES FOR AIMING IN THE DIRECTION OF INVARIANT GRADIENTS†

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A design for constructing control strategies using extremal aiming in the direction of the coinvariant gradients of auxiliary functionals of the Lyapunov–Krasovskii type is presented for problems of controlling hereditary dynamical systems when there is interference. It is proved, under fairly general conditions, that these strategies yield the optimal guaranteed result.

The method of extremal aiming in positional control problems, which dates back to publications by N. N. Krasovskii (see [1–5], for example), has been widely developed in the modern theory of control processes and the theory of differential games. Appropriate extremal aiming designs are used in different problems to prove the existence of optimal solutions and the efficient construction of the resolving control laws using the principle of negative feedback [4–7]. Extremal aiming is used in control procedures with a guide [3, 4] which stabilized the optimal motion and in dynamic methods to solve inverse problems in dynamics [8].

This paper continues the investigation presented in [6, 9–12] of control problems with hereditary information, developing the design of extremal aiming in the direction of quasigradients, proposed earlier in [13, 14] for problems of controlling ordinary differential systems. The problem is formalized within the framework of the game-theoretic approach [4, 5] in combination with a functional treatment of the control process, which is close to that indicated earlier in [9, 15]. Elements of invariant differential calculus [16], non-smooth analysis [17] and results [10, 12, 18] on the development of the theory of generalized (minimax, viscous) solutions of the Hamilton–Jacobi type equations [19, 20] for hereditary systems are used. A similar design was considered in special cases in [11, 12]. A considerable refinement and extension of the results in these papers is given below.

1. BASIC ASSUMPTIONS

Consider a dynamical system described by a differential equation of the form

$$\begin{gathered} \dot{x}[t] = f(t, x[t_*[\cdot]t], u[t], \upsilon[t]), \quad t_* \le t_0 \le t \le T \\ x[t] \in R^n, \quad u[t] \in P \subset R^k, \quad \upsilon[t] \in Q \subset R^m \end{gathered} \tag{1.1}$$

with the initial condition

$$x[t_*[\cdot]t_0] = x_0[t_*[\cdot]t_0] \in C([t_*, t_0], R^n) \tag{1.2}$$

Here t is the time variable, $x[t]$ and $\dot{x}[t] = dx[t]/dt$ are the value of the phase vector and the rate at which it is changing at the current instant of time t, $x[t_*[\cdot]t] = \{x[\tau], t_* \le \tau \le t\}$ is the history of the motion which has been accumulated up to the instant of time t, $u[t]$ is the current action of the control, $\upsilon[t]$ is

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the action of the uncontrolled interference, P and Q are known compacta, t_* and $T(t_* < T)$ are known instants of time, t_0 is the instant when the control process starts and $x_0[t_*[\cdot]t_0]$ is the initial history. Measurable samples of the control and the interference $u[\cdot]$: $[t_0, T] \mapsto P$ and $\upsilon[\cdot]$: $[t_0, T) \mapsto Q$ are permitted. In the case of the initial condition (1.2), the function $x[\cdot] \in C([t_*, T], R^n)$, which is identical with $x_0[t_*[\cdot]t_0]$ in $[t_*, t_0]$, absolutely continuous in $[t_0, T]$ and satisfies Eq. (1.1) for almost all $t \in [t_0, T]$ is the motion of system (1.1). Here, the history of the motion $x[t_*[\cdot]t]$ is the contraction of this function in $[t_*, t]$. We shall call the triplet $\{x[\cdot], u[\cdot], \upsilon[\cdot]\}$ a sample of the control process being considered.

Suppose the quality of the control process is estimated by the characteristic

$$\gamma = \gamma(\{x[\cdot], u[\cdot], \upsilon[\cdot]\}) = \sigma(x[\cdot]) - \int_{t_0}^{T} h(t, x[t_*[\cdot]t], u[t], \upsilon[t])dt \tag{1.3}$$

The purpose of the control is to make the value of this characteristic as small as possible One must take into account here that the actions of the interference are unpredictable and can be very unfavourable.

In relations (1.1) and (1.2), we assume that the function $f = f(t, x[t_*[\cdot]t], u, \upsilon) \in R^n$ and the functional $h = h(t, x[t_*[\cdot]t], u, \upsilon) \in R$ are defined for all $t \in [t_*, T]$, $x[t_*[\cdot]t] \in C([t_*, t], R^n)$, $u \in P$ and $\upsilon \in Q$, continuous in the set of variables $x[t_*[\cdot]t]$, u and υ for any fixed value of t and for any fixed function $x[\cdot] \in C([t_*, T], R^n)$, continuous in the set of variables, t, u and υ and, for any compactum $D \subset C([t_*, T], R^n)$, equipotentially with respect to $x[\cdot] \in D$. The limit

$$\|f(t, x[t_*[\cdot]t], u, \upsilon)\|^2 + h^2(t, x[t_*[\cdot]t], u, \upsilon) \le L^2(t, x[t_*[\cdot]t]) \tag{1.4}$$

is satisfied, where

$$L(t, x[t_*[\cdot]t]) = \left(1 + \max_{t_* \le \tau \le t} \|x[\tau]\|\right)c, \quad c = \text{const} > 0$$

and, for any $s \in R^n$, the equality

$$\begin{aligned} &\min_{u \in P}\max_{\upsilon \in Q}[\langle s, f(t, x[t_*[\cdot]t], u, \upsilon)\rangle - h(t, x[t_*[\cdot]t], u, \upsilon)] = \\ &= \max_{\upsilon \in Q}\min_{u \in P}[\langle s, f(t, x[t_*[\cdot]t], u, \upsilon)\rangle - h(t, x[t_*[\cdot]t], u, \upsilon)] = H(t, x[t_*[\cdot]t], s) \end{aligned} \tag{1.5}$$

holds.

The quantity H, which is defined by the equality, is called the Hamiltonian of system (1.1), (1.3). Henceforth, the symbol $\|\cdot\|$ denotes the Euclidean norm of a vector and $\langle\cdot, \cdot\rangle$ denotes the scalar product of vectors.

As regards the functional $\sigma = \sigma(x[\cdot])$, we assume that it is defined and continuous is $C([t_*, T], R^n)$.

We will denote the set of functions $y[\cdot] \in C([t_*, T], R^n)$, each of which is identical with $x[t_*[\cdot]t]$ in $[t_*, t]$ and is Lipschitzian in $[t, T]$, by the symbol $\text{Lip}(t, x[t_*[\cdot]t])$. The set of functions which, for almost all $X^M(t, x[t_*[\cdot]t])$, satisfy the differential equality

$$\|\dot{y}[\tau]\| \le L(\tau, y[t_*[\cdot]\tau]) + cM$$

is denoted by the symbol $\tau \in [t, T]$.

By virtue of the estimate (1.4), the inclusion

$$x[\cdot] \in X^M(t_0, x_0[t_*[\cdot]t_0]), \quad M \ge 0 \tag{1.6}$$

will hold for any possible sample $\{x[\cdot], u[\cdot], \upsilon[\cdot]\}$ of the control process (1.1)–(1.3).

2. CONTROL STRATEGIES AND THE FUNCTIONAL OF THE OPTIMAL GUARANTEED RESULT

We identify a control strategy with an arbitrary function

$$U = U(t, x[t_*[\cdot]t]) \in P$$

The control process based on a strategy U is accomplished in a scheme which is discrete with respect to time. The subdivision of the time interval $[t_0, T]$

$$\Delta = \{t_i : t_1 = t_0, t_{i+1} > t_i, i = 1, \dots, N, t_{N+1} = T\}$$

is chosen and the following control is formed successively over the steps of this subdivision in the negative feedback circuit

$$u[t] = U(t_i, x[t_*[\cdot]t_i]), \quad t_i \le t < t_{i+1}, \quad i = 1, \dots, N \tag{2.1}$$

The set of all possible samples of the control process, corresponding to a selected strategy U and a subdivision Δ, is denoted by the symbol $S(t_0, x_0[t_*[\cdot]t_0], U, \Delta)$. Actually, this set consists of triplets $\{x[\cdot], u[\cdot], \upsilon[\cdot]\}$ such that $\upsilon[\cdot]$: $[t_0, T) \mapsto Q$ is a measurable function, $u[\cdot]$ is a piecewise constant function of the form (2.1), and $x[\cdot]$: $[t_*, T] \mapsto R^n$ is a continuous function, satisfying condition (1.2), which is absolutely continuous in $[t_0, T]$ and, together with $u[\cdot]$, $\upsilon[\cdot]$, satisfies Eq. (1.1) almost everywhere. Under the assumptions which have been made, the set $S(t_0, x_0[t_*[\cdot]t_0], U, \Delta)$ is non-empty.

Following the principle of a guaranteed result, we define the quantity

$$\Gamma(t_0, x_0[t_*[\cdot]t_0], U, \Delta) = \sup\gamma(S(t_0, x_0[t_*[\cdot]t_0], U, \Delta)) \tag{2.2}$$

where here and henceforth, we use the notation $\sup \rho(A) = \sup \rho(a)$ when $a \in A$.

The optimal guaranteed result OGR of the control will be

$$\varphi(t_0, x_0[t_*[\cdot]t_0]) = \inf_{U, \Delta}\Gamma(t_0, x_0[t_*[\cdot]t_0], U, \Delta) \tag{2.3}$$

According to equality (2.3), the strategy U° is optimal if, for any number $\zeta > 0$, a subdivision Δ is found such that

$$\Gamma(t_0, x_0[t_*[\cdot]t_0], U^\circ, \Delta) \le \varphi(t_0, x_0[t_*[\cdot]t_0]) + \zeta \tag{2.4}$$

We shall also consider the so-called ε-strategies

$$U_\varepsilon = U_\varepsilon(t, x[t_*[\cdot]t]) \in P$$

where $\varepsilon > 0$ is an accuracy parameter (see [5, p. 68]) which is chosen prior to the start of the control process. The optimal strategy will be the ε-strategy U_ε° for which an $\zeta > 0$ and Δ are found for any $\varepsilon > 0$ such that the inequality (2.4) is satisfied (where instead of U°, we write U_ε°).

The value of the OGR depends on the initial position $\{t_0, x_0[t_*[\cdot]t_0]\}$, Consequently, the following OGR functional can be defined

$$\{t \in [t_*, T], x[t_*[\cdot]t] \in C([t_*, t], R^n)\} \mapsto \varphi = \varphi(t, x[t_*[\cdot]t]) \in R \tag{2.5}$$

When $t = T$, this functional satisfies the condition

$$\varphi(T, x[t_*[\cdot]T]) = \sigma(x[\cdot]), \quad x[t_*[\cdot]T] = x[\cdot] \in C([t_*, T], R^n) \tag{2.6}$$

Its lower closure

$$\bar{\varphi}(t, x[t_*[\cdot]t]) = \liminf_{\delta \downarrow 0}\left\{\varphi(t, y[t_*[\cdot]t]) \mid \max_{t_* \le \tau \le t} \|x[\tau] - y[\tau]\| \le \delta\right\} \tag{2.7}$$

possesses the property called u-stability in the theory of differential games [4–6]. In the case being considered, this property can be expressed as follows [12, 18].

Property A. For any $\tau_* \in [t_*, T)$, $y_*[t_*[\cdot]\tau_*] \in C([t_*, \tau_*], R^n)$ and $M \ge 0$, $s \in R^n$, a function $(y[\cdot], z[\cdot]) \in C([t_*, T], R^n \times R)$ exists which is absolutely continuous in $[\tau_*, T]$ and such that

$$y[\cdot] \in X^M(\tau_*, y_*[t_*[\cdot]\tau_*]), \quad z[\tau_*] = \bar{\varphi}(\tau_*, y_*[t_*[\cdot]\tau_*]) \tag{2.8}$$

$$\dot{z}[t] = \langle \dot{y}[t], s\rangle - H(t, y[t_*[\cdot]t], s) \text{ for almost every } t \in [\tau_*, T] \tag{2.9}$$

$$z[t] = \bar{\varphi}(t, y[t_*[\cdot]t]), \quad t \in [\tau_*, T] \tag{2.10}$$

3. AUXILIARY DEFINITIONS

We will introduce the following definitions for functionals of the form

$$\{t \in [t_*, T], w[t_*[\cdot]t] \in C([t_*, t], R^n)\} \mapsto \rho = \rho(t, w[t_*[\cdot]t]) \in R \tag{3.1}$$

Definition 1. We will say that the functional (3.1) is $[t', t'')$-continuous ($[t', t'']$-continuous respectively), where $[t', t''] \subseteq [t_*, T]$, if, first, it is continuous in $w[t_*[\cdot]t] \in C([t_*, t], R^n)$ for any fixed $t \in [t', t'')$ ($t \in [t', t'']$) and, second, it is continuous in t in $[t', t'')$ (in $[t', t'']$ respectively) along any fixed function $w[\cdot] \in C([t_*, T], R^n)$ equipotentially with respect to $w[\cdot] \in D$ for any compactum $D \subset C([t_*, T], R^n)$. The functional ρ is continuous if it is $[t_*, T]$-continuous.

Definition 2. We will say that the functional (3.1) is piecewise-continuous if a finite number, q, of points of discontinuity $t_j \in [t_*, T]$ $(t_1 = t_*,\ t_q = T)$ exists such that it is $[t_j, t_{j+1})$-continuous for any $j = 1, \dots, q-1$.

Definition 3. Functional (3.1) is coinvariantly (ci)-differentiable if, for any $t \in ([t_*, T]$ and $w[t_*[\cdot]t] \in C([t_*, t], R^n)$, $\partial_t\rho = \partial_t\rho(t, w[t_*[\cdot]t] \in R$ and $\nabla\rho = \nabla\rho(t, w[t_*[\cdot]t] \in R^n$ exist such that the equality

$$\begin{aligned}&\rho(t+\delta, y[t_*[\cdot]t+\delta]) - \rho(t, w[t_*[\cdot]t]) = \\ &= \partial_t\rho\delta + \langle \nabla\rho, y[t+\delta] - w[t]\rangle + o_{y[\cdot]}(\delta), \quad 0 < \delta \le T - t\end{aligned} \tag{3.2}$$

holds for all $y[\cdot] \in \mathrm{Lip}(t, w[t_*[\cdot]t])$, where $o_{y[\cdot]}(\delta)$ depends on the choice of $y[\cdot]$, $o_{y[\cdot]}(\delta)/\delta \to 0$ when $\delta \to 0+$.

The quantities $\partial_t\rho$ and $\nabla\rho = \{\nabla_1\rho, \dots, \nabla_n\rho\}$ are called the ci-derivative with respect to t and the ci-gradient of the functional ρ respectively. We shall speak of the function ρ being $[t', t'']$-ci-smooth ($[t', t''] \subseteq [t_*, T]$), if it is $[t', t'']$-continuous, ci-differentiable and its ci-derivative $\partial_t\rho$ and the components $\nabla_k\rho$, $k = 1, \dots, n$ of its ci-gradient $\nabla\rho$ are $[t', t'')$-continuous functionals. We shall say that the functional ρ is ci-smooth if it is $[t_*, T]$-ci-smooth. Details of the technique of the invariant differential calculus of functionals have been described, for example, in [16].

4. THE CASE OF THE CI-SMOOTH FUNCTIONAL FOR THE OPTIMAL GUARANTEED RESULT (OGR)

If the OGR functional φ is ci-smooth then, by virtue of system (1.1), the formula

$$\frac{d}{dt}\varphi(t, x[t_*[\cdot]t]) = \partial_t\varphi + \langle \nabla\varphi, f(t, x[t_*[\cdot]t], u[t], v[t])\rangle \text{ for almost every } t \in [t_0, T] \tag{4.1}$$

holds for its complete derivative (along the motions of this system).

According to what has been described earlier in [18], in the case of a ci-smooth functional φ, the condition for u-stability (property A) is transformed into the differential inequality

$$\partial_t\varphi + H(t, x[t_*[\cdot]t], \nabla\varphi) \le 0, \quad t \in [t_*, T), \quad x[t_*[\cdot]t] \in C([t_*, t], R^n) \tag{4.2}$$

Here, $\partial_t\varphi = \partial_t\varphi(t, x[t_*[\cdot]t])$ and $\nabla\varphi = \nabla\varphi(t, x[t_*[\cdot]t])$ are the ci-derivative with respect to t and the ci-gradient of the functional φ.

Hence, repeating the arguments (see [5, p. 132], for example) used in making smooth estimates of the guaranteed result, taking account of relations (4.1) and (4.2) for the problem being considered, we obtain [12] that, in cases when the functional φ of the OGR turns out to be ci-smooth, the optimal strategy U° can be constructed by aiming in the direction of its ci-gradient $\nabla\varphi$:

$$U^\circ(t, x[t_*[\cdot]t]) = p(t, x[t_*[\cdot]t], s^\circ), \quad s^\circ = \nabla\varphi(t, x[t_*[\cdot]t]) \tag{4.3}$$

where

$$p(t, x[t_*[\cdot]t], s) \in \underset{u \in P}{\arg\min}\{\max_{v \in Q}[\langle s, f(t, x[t_*[\cdot]t], u, v)\rangle - h(t, x[t_*[\cdot]t], u, v)]\} \tag{4.4}$$

Example 1. Suppose the dynamics of a system is described by the equation

$$\begin{gathered} \dot{x}[t] = A(t)x[t] + \int_0^t K(t, \xi)x[\xi]d\xi + u[t] - v[t], \quad t_* = 0 \le t_0 \le t \le T \\ x[t] \in R^n, \quad u[t], v[t] \in R^n : \|u[t]\| \le 1, \quad \|v[t]\| \le 1 \end{gathered} \tag{4.5}$$

and that the quality factor of the control process has the form

$$\gamma = \|x[T]\|^2 - \int_{t_0}^T (\|v[t]\|^2 - \|u(t)\|^2)dt \tag{4.6}$$

Here, $A(t)$ and $K(t, \xi)$ are continuous $n \times n$-matrix functions. According to inequality (1.5), the Hamiltonian of system (4.5), (4.6) is defined by the equality

$$H(t, x[0[\cdot]t], s) = \left\langle A(t)x[t] + \int_0^t K(t, \xi)x[\xi]d\xi, s\right\rangle \tag{4.7}$$

We denote an $n \times n$ matrix function by $\Psi(\tau, t)$, such that $\Psi(\tau, t) = 0$ when $\tau < t$ and $\Psi(t, t)$ is the identity matrix of the identity transformation, and

$$d\Psi(\tau, t)/dt = -\Psi(\tau, t)A(t) - \int_t^\tau \Psi(\tau, \xi)K(\xi, t)d\xi \quad \text{when} \quad \tau > t \tag{4.8}$$

We put

$$\omega(t, x[0[\cdot]t]) = \Psi(T, t)x[t] + \int_0^t\int_t^T \Psi(T, \xi)K(\xi, \eta)x[\eta]d\xi d\eta \tag{4.9}$$

Then, the equality

$$x[T] = \omega(t, x[0[\cdot]t]) + \int_t^T \Psi(T, \eta)(u[\eta] - v[\eta])d\eta \tag{4.10}$$

holds for any possible form $\{x[\cdot], u[\cdot], v[\cdot]\}$ of the process for controlling system (4.5).

Since it is always possible to encounter a situation when $v[\cdot] = u[\cdot]$, we deduce from relations (4.6) and (4.10) that it is impossible here to guarantee anything better than

$$\varphi(t, x[0[\cdot]t]) = \|\omega(t, x[0[\cdot]t])\|^2 \tag{4.11}$$

By virtue of relations (4.8) and (4.9), the functional (4.11) is ci-smooth and, at the same time,

$$\partial_t\varphi = -\left\langle A(t)x[t] + \int_0^t K(t, \xi)x[\xi]d\xi, \nabla\varphi\right\rangle, \quad \nabla\varphi = 2\Psi^{\mathsf{T}}(T, t)\omega(t, x[0[\cdot]t]) \tag{4.12}$$

(the superscript T denotes transposition). It is seen from relations (4.7) and (4.12) that the functional (4.11) satisfies inequality (4.2). Hence, in the case being considered, it is possible to construct strategy (4.3):

$$U^\circ(t, x[0[\cdot]t]) = \begin{cases} -\nabla\varphi/2, & \text{if} \quad \|\nabla\varphi\| \le 2 \\ -\nabla\varphi/\|\nabla\varphi\| & \text{otherwise} \end{cases}$$

For any initial position $\{t_0 \in [0, T], x_0[0[\cdot]t_0] \in C([0, t_0], R^n)\}$ and any number $\zeta > 0$, the control of system (4.5) using the above-mentioned strategy U° enables one to ensure a value

$$\gamma \le \|\omega(t_0, x_0[0[\cdot]t_0])\|^2 + \zeta$$

for the factor (4.6) whatever the permissible form of the interference $\upsilon[\cdot]$. On the other hand, there is no strategy U which would enable one to guarantee a better result in this case.

5. THE GENERAL CASE

In the general case, the function φ of the OGR does not possess suitable smoothness properties. Its ci-derivatives do not exist at all points $\{t, x[t_*[\cdot]t]\}$ and formula (4.3) cannot be used to construct the optimal control strategy. However, following the method proposed in [13, 14] for problems of controlling ordinary differential systems, it is possible to construct an optimal ε-strategy in a fairly general case by replacing the ci-gradient $\nabla\varphi$ in (4.3) (which exist) by a suitable gradient of a necessarily ci-smooth auxiliary functional of the Lyapunov–Krasovskii type.

Thus, suppose D_0 is a compactum from $C([t_*, t_0], R^n)$ such that

$$x_0[t_*[\cdot]t_0] \in D_0 \tag{5.1}$$

Fixing $M \ge 0$, we put

$$X_0 = \{y[\cdot] \in X^M(t_0, x[t_*[\cdot]t_0]) | x[t_*[\cdot]t_0] \in D_0\} \tag{5.2}$$

We now consider the auxiliary functional

$$\{t \in [t_*, T], w[t_*[\cdot]t] \in C([t_*, t], R^n)\} \mapsto \nu_\varepsilon = \nu_\varepsilon(t, w[t_*[\cdot]t]) \in R, \quad \varepsilon > 0$$

We require that the following conditions should be satisfied:
(a) the functional ν_ε is non-negative, continuous and ci-differentiable, and the ci-derivative $\partial_t\nu_\varepsilon$ and the components of the ci-gradient $\nabla\nu_\varepsilon$ of this functional are piecewise-continuous (see definitions 1–3);
(b) the limit $\nu_\varepsilon(t, w[t_*[\cdot]t] \equiv 0) \le \varepsilon$ holds;
(c) for any number $L > 0$ and $\mu > 0$, and $\varepsilon > 0$ exists such that, for any $x[\cdot], y[\cdot] \in X_0$, the inequality $\nu_\varepsilon(T, w[t_*[\cdot]T]) < L$ where $w[\cdot] = x[\cdot] - y[\cdot]$ implies the inequality $|\sigma(x[\cdot]) - \sigma(y[\cdot])| < \mu$;
(d) the inequality

$$\partial_t\nu_\varepsilon + H(t, x[t_*[\cdot]t], \nabla\nu_\varepsilon) - H(t, y[t_*[\cdot]t], \nabla\nu_\varepsilon) \le 0 \tag{5.3}$$

holds for any $t \in [t_*, T]$ and $x[\cdot], y[\cdot] \in X_0$, where

$$\partial_t\nu_\varepsilon = \partial_t\nu_\varepsilon(t, w[t_*[\cdot]t]), \quad \nabla\nu_\varepsilon = \nabla\nu_\varepsilon(t, w[t_*[\cdot]t]), \quad w[\cdot] = x[\cdot] - y[\cdot]$$

The requirement that a functional ν_ε with properties (a)–(d) should exist imposes an additional constraint on the dynamical system being considered. Nevertheless, the class of systems which satisfy this requirement is quite extensive.

For example [11, 12], suppose the Hamiltonian of system (1.1), (1.3), which is defined by equality (1.5), satisfies the Lipschitz condition: a $\lambda > 1$ exist such that, for any $t \in [t_*, T]$, $s \in R^n$ and $x[\cdot], y[\cdot] \in X_0$, the inequality

$$|H(t, x[t_*[\cdot]t], s) - H(t, y[t_*[\cdot]t], s)| \le \lambda(1 + \|s\|)\left(\int_{t_*}^{t} \|w[\tau]\|^2 d\tau + \|w[t]\|^2\right)^{1/2} \tag{5.4}$$

holds, where, as before, $w[\cdot] = x[\cdot] - y[\cdot]$. It is then always possible to take

$$\nu_\varepsilon = \alpha_\varepsilon(t)\beta_\varepsilon, \quad 0 < \varepsilon < \exp\{-2\lambda(T - t_*)\} \tag{5.5}$$

where

$$\alpha_\varepsilon(t) = (\exp\{-2\lambda(t - t_*)\} - \varepsilon)/\varepsilon, \quad \beta_\varepsilon = \left(\varepsilon^4 + 2\lambda\int_{t_*}^{t} \|w[\tau]\|^2 d\tau + \|w[t]\|^2\right)^{1/2}$$

The functional ν_ε is ci-smooth so that it satisfies condition a. Its ci-derivatives are defined by the equalities

$$\partial_t \nu_\varepsilon = -2\lambda \exp\{-2\lambda(t-t_*)\}\beta_\varepsilon/\varepsilon + \lambda\alpha_\varepsilon(t)\|w[t]\|^2/\beta_\varepsilon, \quad \nabla\nu_\varepsilon = \alpha_\varepsilon(t)w[t]/\beta_\varepsilon \tag{5.6}$$

It follows from relations (5.4) and (5.6) that the functional (5.5) satisfies condition d. It can be directly verified that it also satisfies condition b and c.

As a second example of the choice of the auxiliary functional ν_ε, consider the case when $t_* = -\vartheta$, $\vartheta = \text{const} > 0$, $t_0 \geq 0$ and

$$f(t, x[t_*[\cdot]t], u, v) = A(t)x[t] + A_\vartheta(t)x[t-\vartheta] + f(t, u, v), \quad h(t, x[t_*[\cdot]t], u, v) = h(t, u, v)$$

In this case, it is possible to take

$$\nu_\varepsilon = (2\varepsilon)^{-1}\left(\int_{t_*}^{t}\|w[\tau]\|^2 d\tau + \int_{t}^{T}\|\omega(\tau|t, w[t_*[\cdot]t])\|^2 d\tau\right) \tag{5.7}$$

where

$$\omega(\tau|t, w[t_*[\cdot]t]) = \Phi(\tau, t)w[t] + \int_{t}^{t+\vartheta}\Phi(\tau, \xi)A_\vartheta(\xi)w[\xi-\vartheta]d\xi$$

Here, $\Phi(\tau, t)$ is an $n \times n$ matrix function such that $\Phi(\tau, t) = 0$ when $\tau < t$, $\Phi(t, t)$ is the identity matrix and

$$d\Phi(\tau, t)/dt = -\Phi(\tau, t)A(t) - \Phi(\tau, t+\vartheta)A_\vartheta(t+\vartheta) \quad \text{when} \quad \tau > t$$

The functional (5.7) is ci-smooth and

$$\partial_t \nu_\varepsilon = -\langle A(t)w[t] + A_\vartheta(t)w[t-\vartheta], \nabla\nu_\varepsilon\rangle$$

$$\nabla\nu_\varepsilon = \varepsilon^{-1}\int_{t}^{T}\Phi^{\mathrm{T}}(\tau, t)\omega(\tau|t, w[t_*[\cdot]t])d\tau$$

Hence, taking into account the fact that, in the case considered, the Hamiltonian H has the form

$$H(t, x[t_*[\cdot]t], s) = \langle A(t)x[t] + A_\vartheta(t)x[t-\vartheta], s\rangle + \min_{u\in P}\max_{v\in Q}[\langle s, f(t, u, v)\rangle - h(t, u, v)]$$

we obtain the conditions a and b are satisfied in the case of this functional. Since, according to expression (5.7),

$$\nu_\varepsilon(t, w[t_*[\cdot]t] \equiv 0) = 0, \quad \nu_\varepsilon(T, w[t_*[\cdot]T]) = (2\varepsilon)^{-1}\int_{t_*}^{T}\|w[\tau]\|^2 d\tau$$

conditions b and c will also be satisfied.

Note that, in specific problems, a suitable choice of the auxiliary functional ν_ε enables one to simplify considerably the construction of the extremal ε-strategy considered below.

On the basis of the auxiliary functional ν_ε, we now consider the following transformations of the lower closure (2.7) of the functional (2.5) for the OGR

$$\varphi_\varepsilon = \varphi_\varepsilon(t, x[t_*[\cdot]t]) = \min_{y[\cdot]\in X_0}[\bar{\varphi}(t, y[t_*[\cdot]t]) + \nu_\varepsilon(t, w[t_*[\cdot]t])] \tag{5.8}$$

where

$$w[t_*[\cdot]t] = \{w[\tau] = x[\tau] - y[\tau],\ t_* \leq \tau \leq t\}$$

The set X_0 is compact in $C([t_*, T], R^n)$ and the functional $\bar{\varphi} = \bar{\varphi}(t, y[t_*[\cdot]t])$ is semicontinuous from below with respect to $y[t_*[\cdot]t] \in C([t_*, t], R^n)$ for any fixed $t \in ([t_*, T]$ so that a minimum in (5.8) is actually reached. Suppose $y^e[\cdot] \in X_0$ is the minimizing function in (5.8). It depends on the position $\{t, x[t_*[\cdot]t]\}$ and the parameter $\varepsilon > 0$. We put

$$w^e[t_*[\cdot]t] = \{w^e[\tau] = x[\tau] - y^e[\tau], t_* \leq \tau \leq t\}$$

We determine the extremal ε-strategy U_ε^e by aiming in the direction of the ci-gradient $\nabla\nu_\varepsilon(t, w^e[t_*[\cdot]t])$:

$$U_\varepsilon^e(t, x[t_*[\cdot]t]) = p(t, x[t_*[\cdot]t], s^e), \quad s^e = \nabla\nu_\varepsilon(t, w^e[t_*[\cdot]t]) \tag{5.9}$$

Here, the function $p(t, x[t_*[\cdot]t], s)$ satisfies inclusion (4.4).

Theorem 1. For any number $\zeta > 0$, a number $\varepsilon > 0$ and a subdivision Δ of the time interval $[t_0, T]$ exist such that the inequality

$$\Gamma(t_0, x_0[t_*[\cdot]t_0], U_\varepsilon^e, \Delta) \le \bar\varphi(t_0, x_0[t_*[\cdot]t_0]) + \zeta \tag{5.10}$$

will be satisfied for any initial position $\{t_0, x_0[t_*[\cdot]t_0]\}$ which satisfies condition (5.1).

Proof. By virtue of condition *b*, according to equalities (5.8) we have

$$\varphi_\varepsilon(t, x[t_*[\cdot]t]) \le \bar\varphi(t, x[t_*[\cdot]t]) + \varepsilon \tag{5.11}$$

for all $t \in [t_*, T]$ and $x[\cdot] \in X_0$.

We will put

$$K = \max|\sigma(X_0)| < \infty$$

Then, taking relations (2.6) and (2.7) into account, it follows from (5.11), when $t = T$, that

$$\varphi_\varepsilon(T, x[t_*[\cdot]T]) \le K + \varepsilon$$

On the other hand

$$\varphi_\varepsilon(T, x[t_*[\cdot]T]) = \sigma(y^e[\cdot]) + \nu_\varepsilon(T, w^e[t_*[\cdot]T]) \ge -K + \nu_\varepsilon(T, w^e[t_*[\cdot]T])$$

Consequently, the inequality

$$\nu_\varepsilon(T, w^e[t_*[\cdot]T]) \le 2K + \varepsilon$$

holds, by virtue of which, taking into account the non-negativeness of the functional ν_ε and condition *c*, we derive the estimate

$$\varphi_\varepsilon(T, x[t_*[\cdot]T]) \ge \sigma(y^e[\cdot]) \ge \sigma(x[\cdot]) - \mu(\varepsilon), \quad x[\cdot] \in X_0 \tag{5.12}$$

where $\mu(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$.

It is now sufficient to show that

$$\{x[\cdot], u[\cdot], \upsilon[\cdot]\} \in S(t_0, x_0[t_*[\cdot]t_0], U_\varepsilon^e, \Delta) \tag{5.13}$$

will satisfy the inequalities

$$\begin{aligned} &\varphi_\varepsilon(t_{i+1}, x[t_*[\cdot]t_{i+1}]) - \int_{t_i}^{t_{i+1}} h(t, x[t_*[\cdot]t], u[t], \upsilon[t])dt \le \\ &\le \varphi_\varepsilon(t_i, x[t_*[\cdot]t_i]) + \eta(\delta)(t_{i+1} - t_i), \quad i = 1, \ldots, N; \quad \delta = \max_{i=1,\ldots,N}(t_{i+1} - t_i) \end{aligned} \tag{5.14}$$

for any of the possible samples.

Here Δ is the subdivision of the time interval $[t_0, T]$ in which all the points t_j of possible discontinuities of the ci-derivative $\partial_t\nu_\varepsilon$ and the components of the ci-gradient $\nabla\nu_\varepsilon$ of the auxiliary functional ν_ε are included, t_i are the points of this subdivision, δ is its diameter and $\eta(\delta) \downarrow 0$ when $\delta \downarrow 0$.

By virtue of inequalities (5.14), we actually have

$$\varphi_\varepsilon(T, x[t_*[\cdot]T]) - \int_{t_0}^{T} h(t, x[t_*[\cdot]t], u[t], \upsilon[t])dt \le \varphi_\varepsilon(t_0, x_0[t_*[\cdot]t_0]) + \eta(\delta)(T - t_0)$$

whence, taking into account equality (1.3), inclusion (1.6) together with the notation (5.2), inequality (5.11) and estimate (5.12), we derive the inequality

$$\bar{\varphi}(t_0, x_0[t_*[\cdot]t_0]) + \varepsilon + \mu(\varepsilon) + \eta(\delta)(T - t_0) \geq \gamma(\{x[\cdot], u[\cdot], v[\cdot]\})$$

It will be satisfied for all the samples (5.13) Hence, by choosing $\varepsilon > 0$ and Δ from the condition

$$\varepsilon + \mu(\varepsilon) + \eta(\delta)(T - t_0) \leq \zeta$$

we obtain the required inequality (5.10) from this in accordance with definition (2.2).

We will now prove inequalities (5.14). Fixing the samples (5.13) and $i = 1, \ldots, N$, we put

$$g^x[t] = \{t, x[t_*[\cdot]t]\}, \quad \tau_* = t_i, \quad \tau^* = t_{i+1}$$

It is required to show that

$$\varphi_\varepsilon(g^x[\tau^*]) = \int_{\tau_*}^{\tau^*} h(g^x[t], u[t], v[t])dt \leq \varphi_\varepsilon(g^x[\tau_*]) + \eta(\delta)(\tau^* - \tau_*)$$

where $\eta(\delta) \downarrow 0$ when $\delta \downarrow 0$ and $\eta(\delta)$ does not depend on the chosen sample (5.13). By the construction (5.8) and (5.9), for $t = \tau_*$, we put

$$y_*[\cdot] = y^e[\cdot], \quad g_*^y = \{\tau_*, y_*[t_*[\cdot]\tau_*]\}, \quad g_*^w = \{\tau_*, w^e[t_*[\cdot]\tau_*]\}$$

$$s = s^e = \nabla \nu_\varepsilon(g_*^w), \quad u_*^e = U_\varepsilon^e(g^x[\tau_*]) = p(g^x[\tau_*], s)$$

We have

$$\varphi_\varepsilon(g^x[\tau_*]) = \bar{\varphi}(g_*^y) + \nu_\varepsilon(g_*^w) \tag{5.15}$$

$$u[t] = u_*^e, \quad \dot{x}[t] = f(g^x[t], u[t], v[t]) \quad \text{for almost every} \quad t \in [\tau_*, \tau^*) \tag{5.16}$$

At the same time, according to equality (1.5) and inclusion (4.4), the relations

$$\begin{aligned} H(g^x[\tau_*], s) &= \max_{v \in Q}[\langle s, f(g^x[\tau_*], u_*^e, v)\rangle - h(g^x[\tau_*], u_*^e, v)] \geq \\ &\geq \langle s, f(g^x[\tau_*], u_*^e, v[t])\rangle - h(g^x[\tau_*], u_*^e, v[t]) \end{aligned} \tag{5.17}$$

hold.

On the basis of the property of u-stability (property A), we take the function $(y[\cdot], z[\cdot])$ which satisfies conditions (2.8)–(2.10) and then use the notation

$$g^y[t] = \{t, y[t_*[\cdot]t]\}, \quad w[\cdot] = x[\cdot] - y[\cdot], \quad g^w[t] = \{t, w[t_*[\cdot]t]\}$$

Note that $g^y[\tau_*] = g_*^y$, $g^w[\tau_*] = g_*^w$ here. From relations (2.8), (2.10) and (5.15), taking into account the notation adopted, we derive

$$\varphi_\varepsilon(g^x[\tau_*]) = \bar{\varphi}(g^y[\tau^*]) - \int_{\tau_*}^{\tau^*} \dot{z}[t]dt + \nu_\varepsilon(g_*^w) \tag{5.18}$$

Since $y[\cdot] \in X^M$ $(g_*^y = \{\tau_*, y_*[t_*[\cdot]\tau_*]\})$ and $y_*[\cdot] \in X_0$, then $y[\cdot] \in X_0$ the inequality

$$\bar{\varphi}(g^y[\tau^*]) + \nu_\varepsilon(g^w[\tau^*]) \geq \varphi_\varepsilon(g^x[\tau^*])$$

will therefore be satisfied in accordance with equality (5.8).

From this and from relation (5.18), we conclude that

$$\varphi_\varepsilon(g^x[\tau^*]) - \int_{\tau_*}^{\tau^*} h(g^x[t], u[t], \upsilon[t])dt \le \varphi_\varepsilon(g^x[\tau_*]) + \theta \tag{5.19}$$

where

$$\theta = \nu_\varepsilon(g^w[\tau^*]) - \nu_\varepsilon(g^w_*) + \int_{\tau_*}^{\tau^*} [\dot{z}[t] - h(g^x[t], u[t], \upsilon[t])]dt \tag{5.20}$$

We will now estimate θ. Consider the function

$$\nu[t] = \nu_\varepsilon(g^w[t]) = \nu_\varepsilon(t, w[t_*[\cdot]t]), \quad t \in [\tau_*, \tau^*]$$

By virtue of condition a, the functional ν_ε is $[\tau_*, \tau^*]$-ci-smooth. Since $x[\cdot], y[\cdot] \in X_0$, then $w[\cdot] \in \mathrm{Lip}(g^w[t_0])$. Hence, the function $\nu[t]$ is absolutely continuous and, on taking account of relation (3.2) (when $\rho = \nu_\varepsilon$), the equalities

$$\nu_\varepsilon(g^w[\tau^*]) - \nu_\varepsilon(g^w_*) = \int_{\tau_*}^{\tau^*} \dot{\nu}[t]dt = \int_{\tau_*}^{\tau^*} [\partial_t \nu_\varepsilon(g^w[t]) + \langle \nabla\nu_\varepsilon(g^w[t]), \dot{x}[t] - \dot{y}[t]\rangle]dt$$

hold.

Taking this and relations (2.9) and (5.16) into account, the quantity (5.20) can be represented in the following form

$$\theta = \int_{\tau_*}^{\tau^*} [\partial_t \nu_\varepsilon(g^w[t]) + \langle \nabla\nu_\varepsilon(g^w[t]), f(g^x[t], u^e_*, \upsilon[t])\rangle - h(g^x[t], u^e_*, \upsilon[t]) -$$
$$- H(g^y[t], s) - \langle \dot{y}[t], \nabla\nu_\varepsilon(g^w[t]) - s\rangle]dt$$

Hence if we take into account the equality

$$s = \nabla\nu_\varepsilon(g^w_*) = \nabla\nu_\varepsilon(g^w[\tau_*])$$

and the continuity properties of the quantities f, h, $\partial_t\nu_\varepsilon$, $\nabla\nu_\varepsilon$, we obtain the estimate

$$\theta \le \int_{\tau_*}^{\tau^*} [\partial_t \nu_\varepsilon(g^w[\tau_*]) + \langle s, f(g^x[\tau_*], u^e_*, \upsilon[t])\rangle - h(g^x[\tau_*], u^e_*, \upsilon[t]) - H(g^y[\tau_*], s)]dt +$$
$$+ \eta(\delta)(\tau^* - \tau_*), \quad \eta(\delta) \downarrow 0 \quad \text{when} \quad \delta \downarrow 0 \tag{5.21}$$

Since, by virtue of relations (1.6) and (5.2), the functions $x[\cdot]$ and $y[\cdot]$ are always contained in the compactum X_0, the same infinitesimal $\eta(\delta)$ can be taken here for all possible samples (5.13). Taking into account the fact that $s = \nabla\nu_\varepsilon(g^w[\tau_*])$ and using inequality (5.3), we obtain the estimate

$$\theta \le \eta(\delta)(\tau^* - \tau_*)$$

from inequalities (5.17) and (5.21).

This estimate completes the proof of relation (5.14) and, together with it, also the whole theorem. According to definition (2.3) of the magnitude of the OGR, it follows from this theorem that

$$\varphi(t_0, x_0[t_*[\cdot]t_0]) \le \bar{\varphi}(t_0, x_0[t_*[\cdot]t_0])$$

On the other hand, since the functional $\bar{\varphi}$ is the lower closure (2.7) of the functional φ, we have

$$\varphi(t_0, x_0[t_*[\cdot]t_0]) \ge \bar{\varphi}(t_0, x_0[t_*[\cdot]t_0]), \quad t_0 \in [t_*, T], \quad x_0[t_*[\cdot]t_0] \in C([t_*, t_0], R^n)$$

Hence, in the case of the assumptions being considered, $\varphi = \bar{\varphi}$, and, consequently, the extremal ε-strategy U_ε^e is the optimal strategy.

Remark 1. The counter-problem of searching for the interference strategy V° (or V_ε°) which guarantees that the factor (1.3) has the greatest possible value can be treated in a similar manner.

Example 2. Suppose the dynamics of the control system, which has two-dimensional phase vector $x = \{x_1, x_2\} \in R^2$, is described by the equations

$$\dot{x}_1[t] = \int_0^t \ln(1+t+\tau)x_2[\tau]d\tau + u[t], \quad \dot{x}_2[t] = \upsilon[t]$$
$$0 \le t^0 \le t \le T, \quad \|u[t]\| \le a, \quad \|\upsilon[t]\| \le b \tag{5.22}$$

with the initial condition

$$x[\tau] = \{x_1[\tau], x_2[\tau]\} = \{x_1^0[\tau], x_2^0[\tau]\} = x^0[\tau], \quad 0 \le \tau \le t^0 \tag{5.23}$$

The quality factor of the control process is given in the form

$$\gamma = |x_1[T] + x_2[t^*]| - \int_{t^0}^T \sqrt{a^2 - u^2[t]}dt \tag{5.24}$$

The constants $a > 0$ and $b > 0$, the initial instants of time $t^0 \in [0, T)$ and the initial history $x^0[0[\cdot]t^0] \in C([0, t^0], R^2)$, the intermediate instant of time $t^* \in [0, T)$ and the terminal instant $T > 0$ are assumed to be known.
The Hamiltonian of system (5.22), (5.24) has the form

$$H(t, x[0[\cdot]t], s) = s_1\int_0^t \ln(1+t+\tau)x_2[\tau]d\tau - a\sqrt{1+s_1^2} + b|s_2| \tag{5.25}$$

We now use the notation

$$\theta(t,\tau) = \int_t^T \ln(1+\tau+\xi)d\xi, \quad \omega_1(t) = \int_t^T\int_t^\xi \ln(1+\xi+\tau)d\tau d\xi$$

$$\omega_2(t) = \begin{cases} \omega_1(t)+1 & \text{when} \quad t < t^*, \\ \omega_1(t) & \text{when} \quad t \ge t^*, \end{cases} \qquad \omega_3(t) = b\int_t^T \omega_2(\tau)d\tau$$

$$\rho(t, x[0[\cdot]t]) = x_1[t] + \omega_1(t)x_2[t] + \int_0^t \theta(t,\tau)x_2[\tau]d\tau + \begin{cases} x_2[t] & \text{when} \quad t < t^* \\ x_2[t^*] & \text{when} \quad t \ge t^* \end{cases}$$

Note that the functional $\rho = \rho(t, x[0[\cdot]t])$ is continuous and ci-differentiable. We will calculate its ci-derivatives. We have

$$\partial_t\rho = -\int_0^t \ln(1+t+\tau)x_2[\tau]d\tau, \quad \nabla\rho = \{\nabla_1\rho, \nabla_2\rho\} = \{1, \omega_2(t)\} \tag{5.26}$$

It can be seen that the quantities $\partial_t\rho$ and $\nabla_1\rho$ are continuous and that the quantity $\nabla_2\rho$ is piecewise-continuous with a point of discontinuity t^*. We also note that

$$\rho(t, x[0[\cdot]t] \equiv 0) = 0, \quad \rho(T, x[0[\cdot]T]) = x_1[T] + x_2[t^*]$$

Whatever the permissible forms $u[\cdot]: [t^0, T) \mapsto [-a, a]$ of the control and $\upsilon[\cdot]: [t^0, T) \mapsto [-b, b]$ of the interference, the equality

$$x_1[T] + x_2[t^*] = \rho(t^0, x^0[0[\cdot]t^0]) + \int_{t^0}^T (\omega_2(\tau)\upsilon[\tau] + u[\tau])d\tau$$

holds for the corresponding form $x[\cdot] = \{x_1[\cdot], x_2[\cdot]\}: [0, T] \mapsto R^2$ of motion of system (5.22) in the case of initial condition (5.23).

Starting from this equality, for example, by the method from [9] we obtain the following expression for the OGR functional of the control problem (5.22), (5.24)

$$\begin{aligned}\varphi(t, x[0[\cdot]t]) &= \max_{|l| \le 1}[\rho(t, x[0[\cdot]t])l + \omega_3(t)|l| - (T-t)a\sqrt{1+l^2}] = \\ &= \max_{|l| \le 1}[\rho(t, x[0[\cdot]t])l + \bar{\psi}(t, l)]\end{aligned} \tag{5.27}$$

Here, $\bar{\psi}(t, l)$ is the envelope of the function which is convex from above

$$\psi(t, l) = \omega_3(t)|l| - (T-t)a\sqrt{1+l^2}$$

in the set $\{l \in R : |l| \le 1\}$, that is,

$$\bar{\psi}(t, l) = \begin{cases}\psi(t, l_*), & \text{if} \quad |l| \le l_* \\ \psi(t, l), & \text{if} \quad |l| > l_*\end{cases}$$

where

$$l_* = \begin{cases}\omega_3(t)((T-t)^2a^2 - \omega_3^2(t))^{-1/2}, & \text{if} \quad 2\omega_3^2(t) < (T-t)^2a^2 \\ 1 \quad \text{otherwise}\end{cases}$$

In order to construct the corresponding optimal control strategy, we make use of the extremal aiming design (5.8), (5.9) and put

$$D_0 = \{x[\tau] = x^0[\tau] + f,\ 0 \le \tau \le t^0 \,|\, \|f\| \le M_0\}$$

where $M_0 > 0$ is a sufficiently large number. At the same time, we put $M > M_0$ in equality (5.2). We take

$$\nu_\varepsilon(t, w[0[\cdot]t]) = (2\varepsilon)^{-1}\rho^2(t, w[0[\cdot]t])$$

as the auxiliary functional.

By virtue of the properties of the functional ρ noted above, the functional ν_ε satisfies requirements *a–c*. When account is taken of equalities (5.26), its ci-derivatives are defined by the equalities

$$\partial_t\nu_\varepsilon = \partial_t\nu_\varepsilon(t, w[0[\cdot]t]) = -\varepsilon^{-1}\rho(t, w[0[\cdot]t])\int_0^t \ln(1+t+\tau)w_2[\tau]d\tau$$

$$\nabla\nu_\varepsilon = \nabla\nu_\varepsilon(t, w[0[\cdot]t]) = \varepsilon^{-1}\rho(t, w[0[\cdot]t])\{1, \omega_2(t)\}$$

It is seen from this and from expression (5.25) that condition *d* is satisfied in the case of this functional. On carrying out the calculations, we obtain

$$s^e = \nabla\nu_\varepsilon(t, w^e[0[\cdot]t]) = \{l_u, \omega_2(t)l_u\}$$

$$l_u \in \operatorname*{argmax}_{|l| \le 1}[\rho(t, x[0[\cdot]t])l + \bar{\psi}(t, l) - \varepsilon l^2/2]$$

Consequently,

$$U_\varepsilon^e(t, x[0[\cdot]t]) = -al_u/\sqrt{1+l_u^2} \tag{5.28}$$

According to Theorem 1, the ε-strategy U_ε^e which has been constructed is optimal for control problem (5.22)–(5.24).

The result of the modelling of the control process of system (5.22) on the basis of the ε-strategy (5.28) in a pair with an interference strategy $V = b\cos(20tx_1[t]x_2[t])$ taken at random for the case of the following initial data

$$T = 3, \quad t^* = 2, \quad t^0 = 0.5, \quad a = 5, \quad b = 2$$

$$x_1^0[\tau] = \cos 20\tau + 0.5\sin 10\tau, \quad x_2^0[\tau] = \sin 20\tau + 0.5\cos 10\tau \quad \text{when} \quad 0 \le \tau \le 0.5$$

is shown in Fig. 1.

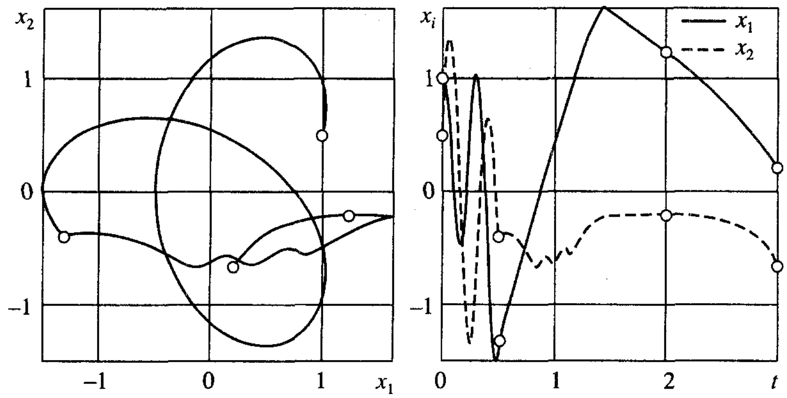

Fig. 1

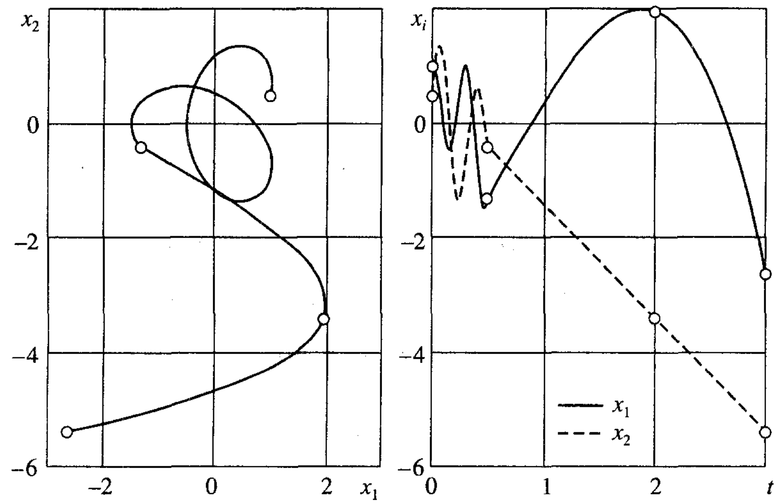

Fig. 2

The value of the accuracy parameter was chosen as $\varepsilon = 0.001$. The action of the strategies was accomplished by uniform subdivision of the time interval [0.5, 3] with a step size $\delta = 0.001$. The a priori calculated magnitude of the OGR was

$$\varphi \approx -2.7843$$

The value of the quality factor realized was

$$\gamma \approx |0.220 + (-0.220)| - 11.251 = -11.251 < \varphi$$

The result of the modelling of the control process using the same ε-strategy U_ε^e under the same conditions but in a pair with a counter-optimal interference ε-strategy, which, in the case being considered, can be defined as follows.

$$V_\varepsilon^\mathrm{o}(t, x[0[\cdot]t]) = \operatorname{sign}(\omega_2(t)l_\upsilon)b$$

$$l_\upsilon \in \underset{|l| \le 1}{\operatorname{argmax}}[\rho(t, x[0[\cdot]t])l + \psi(t, l) + \varepsilon l^2/2]$$

is shown in Fig. 2.

The value of the quality factor obtained was

$$\gamma \approx |(-2.647) + (-3.402)| - 8.839 = -2.790 \approx \varphi$$

In conclusion, we note that transformation (5.8) is similar to the "smoothing" transformations, by means of which quasigradients [13] and also proximal gradients (see [17], for example) of non-smooth functions are determined. Inequalities of the form of (5.3) play an important role in the theory of the generalized solutions of equations of the Hamilton–Jacobi type when proving the uniqueness of the solution (see [18–20], for example). In the case being considered of systems with an aftereffect, the functional ν_ε is a suitable analogue of the auxiliary functions which are used in these constructions.

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